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NOTE ON THE STONELEY WAVES IN ELASTIC MEDIA WITH LONG-RANGE COHESION FORCES†

J. L. NOWINSKI‡
University of Delaware, Newark, DE, 19711, U.S.A.

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Abstract—Stoneley waves at the interface of two nonlocal media are investigated. Using the nonlocal constitutive equations, the problem is solved in the Fourier transform space. The nonlocal elastic moduli appearing in the governing equations are determined by appeal to the atomic lattice dynamics. The frequency equation obtained turns out to be of the form similar to the form of the equation derived in the conventional theory, but predicts a dispersion of waves. For very short waves, the speed of Stoneley wave decreases to about 0.64 of its value for long waves.

1. INTRODUCTORY REMARKS

The last two decades have witnessed the inauguration of a novel theory of material bodies, named the nonlocal mechanics. This was done primarily due to the efforts of Edelen[1], Eringen[2], Green and Rivlin[3], Kroener[4], and Kunin[5], and involved elastic, plastic[6] and liquid[7] media.

The nonlocal mechanics ascribes to bodies a quasi-continuous structure, in the sense that, apart from the ideal continuity, the actual discreteness of matter is to some extent taken into account. This objective is achieved through the replacement of the central postulate of the classical theories stating that the internal particle-to-particle interactions represent contact or zero-range forces by a broader and more realistic assumption (since laboratory experiments detect interactions often reaching the eleventh atomic neighbor), that the latter have a long range; that is, are actually *nonlocal*.

It is of interest to note that applications of the nonlocal theory to concrete problems often lead to impressive agreements with the data of experiments and observations (e.g. [8, 9]). And so, contrary to the predictions of the classical theory of elasticity and in harmony with the experimental evidence, the nonlocal theory concludes that the elastic waves in an infinite space, as well as the surface Rayleigh waves, are dispersive. Likewise, realistic conclusions are drawn with regard to the mechanics of fracture, secondary flow patterns in pipes and others.

In consideration of all these facts, it seems not out of place to inquire about the predictions of the nonlocal theory with regard to another well-known phenomenon of wave mechanics; that is, the Stoneley waves. As commonly known, these are the elastic waves propagating along the surface of separation of two solid half-infinite media; their amplitude being a maximum at the surface itself.

Using the nonlocal constitutive equations, we first apply the Fourier exponential transformation to the governing equations and, by appeal to the dispersion equation derived in the atomic lattice dynamics, arrive at the expressions for the nonlocal elastic moduli. Solution obtained in terms of the displacements enables one to formulate explicitly the boundary conditions, and consequently to arrive at the frequency equation of the problem displaying the dispersion of waves. A few general remarks shed some additional light on the nonlocal aspects of the problem.

† Prepared with partial support of the University of Delaware.
‡ H. Fletcher Brown Professor Emeritus, Department of Mechanical and Aerospace Engineering, University of Delaware, Newark, DE 19716, U.S.A.

2. GENERAL EQUATIONS

Let the plane $x_2 = 0$ of a Cartesian rectangular coordinate system x_1, x_2, x_3 coincide with the plane surface of separation of two half-infinite homogeneous and isotropic elastic media (Fig. 1), with the densities and Lamé constants ρ, μ, λ (medium I, $x_2 > 0$) and ρ^*, μ^*, λ^* (medium II, $x_2 < 0$), respectively. Let along the plane $x_2 = 0$ propagate a plane wave in the direction of the x_1 axis, leaving the media in the state of plane strain.

In standard notation the equations of motion are

$$\tau_{11,1} + \tau_{12,2} = \rho\ddot{u}_1, \qquad \tau_{12,1} + \tau_{22,2} = \rho\ddot{u}_2, \tag{2.1}$$

for the medium I, with analogous equations for the medium II. In what follows, it suffices to discuss the medium I.

Under the assumption that the medium is nonlocal, the stress components may be represented in the Kroener–Eringen form (proposed somewhat intuitively by Kroener, and derived rigorously on the basis of the general theory of constitutive equations by Eringen[2])

$$\tau_{ij}(x, t) = \int_V [2\mu'(\,|\,x' - x\,|\,)e_{ij}(x', t) + \lambda'(\,|\,x' - x\,|\,)e_{kk}(x', t)\,\mathrm{d}x_1'\,\mathrm{d}x_2'\,\mathrm{d}x_3'. \tag{2.2}$$

Here $|\,x' - x\,| \equiv |\,x_1' - x_1\,|, |\,x_2' - x_2\,|, |\,x_3' - x_3\,|$, V represents the infinite space, μ' and λ' are the nonlocal moduli, $e_{ij}(i, j = 1, 2, 3)$ is the linear strain tensor, x is the point of observation, and the prime denotes a generic point of the medium. On account of the theorem of Edelen ([10], Appendix), for the nonlocal moduli tending to zero at infinity, the integrand in (2.2) becomes independent of the variable x_3', and the volume V may be replaced by the area $A(-\infty < x_1 < \infty, 0 \le x_2 < \infty)$. The stress components then become

$$\begin{aligned}
\tau_{11} &= \int_A [(2\mu' + \lambda')u'_{1,1} + \lambda' u'_{2,2}]\,\mathrm{d}x_1'\,\mathrm{d}x_2',\\
\tau_{22} &= \int_A [(2\mu' + \lambda')u'_{2,2} + \lambda' u'_{1,1}]\,\mathrm{d}x_1'\,\mathrm{d}x_2',\\
\tau_{12} &= \int_A \mu'(u'_{1,2} + u'_{2,1})\,\mathrm{d}x_1'\,\mathrm{d}x_2',
\end{aligned} \tag{2.3}$$

where $\mu' \equiv \mu(|\,x_1' - x_1\,|, |\,x_2' - x_2\,|)$, and $u_1' \equiv u_1(x_1', x_2', t)$, for example.

Modeling the procedure on that utilized in [11] with regard to the Rayleigh waves, we apply the Fourier exponential transformation

$$\bar{u}(k, x_2, \omega) = \int_{-\infty}^{-\infty}\int_{-\infty}^{\infty} u(x_1, x_2, t)e^{i(kx_1 + \omega t)}\,\mathrm{d}x_1\,\mathrm{d}t \tag{2.4}$$

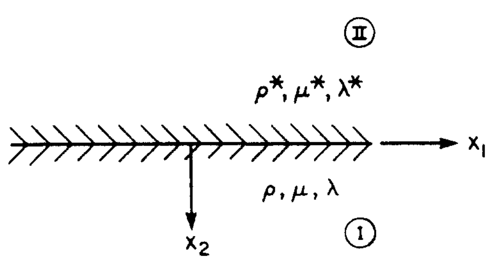

Fig. 1. Geometry of the problem.

to eqns (2.1) and (2.3), and after lengthy computations, similar to those that lead from eqns (2.3) to (2.5) in [10] (this involves the substitution $\eta = x_1' - x_1$, and the change of order of integration), arrive at the following equations:

$$-ik\bar{\tau}_{11}(k, x_2, \omega) + \bar{\tau}_{12,2}(k, x_2, \omega) + \rho\omega^2\bar{u}_1(k, x_2, \omega) = 0, \tag{2.5a}$$

$$-ik\bar{\tau}_{12}(k, x_2, \omega) + \bar{\tau}_{22,2}(k, x_2, \omega) + \rho\omega^2\bar{u}_2(k, x_2, \omega) = 0, \tag{2.5b}$$

and

$$\bar{\tau}_{11}(k, x_2, \omega) = \int_0^\infty \{-ik[2\bar{\mu}'(k, |x_2' - x_2|) + \bar{\lambda}'(k, |x_2' - x_2|)]\bar{u}_1'(k, x_2', \omega) + \bar{\lambda}'(k, |x_2' - x_2|)\bar{u}_{2,2}'(k, x_2', \omega)\}\, dx_2', \tag{2.6a}$$

$$\bar{\tau}_{22}(k, x_2, \omega) = \int_0^\infty \{[2\bar{\mu}'(k, |x_2' - x_2|) + \bar{\lambda}(k, |x_2' - x_2|)]\bar{u}_{2,2}'(k, x_2', \omega) - ik\bar{\lambda}'(k, |x_2' - x_2|)\bar{u}_1'(k, x_2', \omega)\}\, dx_2', \tag{2.6b}$$

$$\bar{\tau}_{12}(k, x_2, \omega) = \int_0^\infty \bar{\mu}'(k, |x_2' - x_2|)[\bar{u}_{1,2}'(k, x_2', \omega) - ik\bar{u}_2'(k, x_2', \omega)]\, dx_2'. \tag{2.6c}$$

We now recall (see [11] and [12]) that by identifying the dispersion equation obtained in the nonlocal theory of elastic waves propagating in an infinite space with the corresponding equation provided by the Born–von Kármán lattice dynamics, one arrives at the following relations for the transforms of the associated nonlocal moduli†:

$$\epsilon(k) \equiv \frac{\bar{\lambda}(k)}{\lambda} = \frac{\bar{\mu}(k)}{\mu} = \frac{\bar{\mu}(k) + \bar{\lambda}(k)}{\lambda + \mu} = \frac{\sin^2\left(\frac{ka}{2}\right)}{\left(\frac{ka}{2}\right)^2}, \tag{2.7}$$

where a is the atomic spacing.‡

As is known, the variation of the argument ka is confined to the so-called first Brillouin zone, $0 \le |ka| \le \Pi$, one of the features of which is that it removes the ambiguity in the wavelength associated with the given frequency. Since the wavelength is $\Lambda = 2\Pi/k$, the latter may consequently vary from $\Lambda = \infty$ to the acceptable minimum $\Lambda = 2a$. Referring now to the relations (2.7), we represent the moduli associated with the problem under discussion in the separable form,

$$\begin{aligned}\bar{\mu}'(k, |x_2' - x_2|) &= \bar{\mu}(k)\delta_n(|x_2' - x_2|),\\ \bar{\lambda}'(k, |x_2' - x_2|) &= \bar{\lambda}(k)\delta_n(|x_2' - x_2|),\end{aligned} \tag{2.8}$$

where δ_n denotes the n-th term of an appropriately selected δ-sequence, whose limit for $n \to \infty$ is the Dirac-delta function $\delta(x_2' - x_2)$. Such a decision seems rather realistic inasmuch as, even for particle interactions reaching the tenth closest neighbor, the radius of interactions remains of the order of 10^{-7} cm. With respect to the selected sequence, we adopt the simplifying assumption that for a sufficiently large n, the terms of the sequence enjoy the shifting property characteristic of the Dirac function. Sub-

† It is assumed that the particle interactions involve closest neighbors. For a more general approach see [13].

‡ Clearly an inverse transformation of eqns (2.7) for the long-wave case ($k \to 0$) in combination with the identity $\int_{-\infty}^{\infty} e^{ikx_1}\, dk = 2\Pi\delta(x_1)$ yields, e.g. $\mu'(|x_1' - x_1|) = \mu\delta(x_1' - x_1)$, so that eqn (2.2) becomes the well-known equation of the classical theory.

stitution of eqns (2.8) into eqns (2.6) now yields

$$\bar{\tau}_{11}(k, x_2, \omega) = -ik[2\bar{\mu}(k) + \bar{\lambda}(k)]\bar{u}_1(k, x_2, \omega) + \bar{\lambda}(k)\bar{u}_{2,2}(k, x_2, \omega), \quad (2.9a)$$

$$\bar{\tau}_{22}(k, x_2, \omega) = [2\bar{\mu}(k) + \bar{\lambda}(k)]\bar{u}_{2,2}(k, x_2, \omega) - ik\bar{\lambda}(k)\bar{u}_1(k, x_2, \omega), \quad (2.9b)$$

$$\bar{\tau}_{12}(k, x_2, \omega) = \bar{\mu}(k)[\bar{u}_{1,2}(k, x_2, \omega) - ik\bar{u}_2(k, x_2, \omega)]. \quad (2.9c)$$

We insert the equations above into the eqns (2.5), arriving at the system of two coupled equations:

$$\bar{\mu}\bar{u}_{1,22} - ik(\bar{\lambda} + \bar{\mu})u_{2,2} + [\rho\omega^2 - k^2(2\bar{\mu} + \bar{\lambda})]\bar{u}_1 = 0, \quad (2.10a)$$

$$-ik(\bar{\mu} + \bar{\lambda})\bar{u}_{1,2} + (2\bar{\mu} + \bar{\lambda})\bar{u}_{2,22} + [\rho\omega^2 - k^2\bar{\mu}]\bar{u}_2 = 0, \quad (2.10b)$$

whose solution does not present difficulties. We have

$$\begin{aligned} \bar{u}_1 &= A_1 e^{-\alpha_1 x_2} + A_2 e^{-\alpha_2 x_2}, \\ \bar{u}_2 &= A_1\sigma_1 e^{-\alpha_1 x_2} + A_2\sigma_2 e^{-\alpha_2 x_2}; \end{aligned} \quad (2.11)$$

$$\begin{aligned} \bar{\tau}_{22} &= -A[(2\bar{\mu} + \bar{\lambda})\alpha_1\sigma_1 - ik\bar{\lambda}] - A_2[(2\bar{\mu} + \bar{\lambda})\alpha_2\sigma_2 - ik\bar{\lambda}], \\ \bar{\tau}_{12} &= -\bar{\mu}[A_1(\alpha_1 + ik\sigma_1) + A_2(\alpha_2 + ik\sigma_2)], \end{aligned} \quad (2.12)$$

where

$$\begin{aligned} &\alpha_1 = \left(1 - \frac{c^2}{\bar{c}_L^2}\right)^{1/2}, \quad \alpha_2 = \left(1 - \frac{c^2}{\bar{c}_T^2}\right)^{1/2}, \\ &\sigma_1 = -i\alpha_1/k, \quad \alpha_2 = -ik/\alpha_2, \\ &c = \omega/k, \quad \bar{c}_L^2 = (2\bar{\mu} + \bar{\lambda})/\rho, \quad \bar{c}_T^2 = \bar{\mu}/\rho. \end{aligned} \quad (2.13)$$

3. BOUNDARY CONDITIONS

Let us now imagine that the half spaces under consideration are in "welded" contact along the interface $x_2 = 0$, so that there is no slipping, and the corresponding stress components are equal:

$$\begin{aligned} \bar{u}_1 &= \bar{u}_1^*, \quad \bar{u}_2 = \bar{u}_2^*, \\ \bar{\tau}_{22} &= \bar{\tau}_{22}^*, \quad \bar{\tau}_{12} = \tau_{12}^*, \end{aligned} \quad (3.1)$$

at $x_2 = 0$ for any x_1 and any t. A nontrivial solution of the system (3.1) of four linear homogeneous equations for four unknowns A_1, A_2, A_1^* and A_2^*, implies that the determinant of the system should vanish. This leads to the following equation for the velocity c of the Stoneley waves:

$$\begin{aligned} c^4[(1 - \gamma)^2 - (\alpha_1^* + \gamma\alpha_1)(\alpha_2^* + \gamma\alpha_2)] + 2Kc^2[\alpha_1^*\alpha_2^* - \gamma\alpha_1\alpha_2 \\ - 1 + \gamma] + K^2(\alpha_1\alpha_2 - 1)(\alpha_1^*\alpha_2^* - 1) = 0, \end{aligned} \quad (3.2)$$

where

$$\begin{aligned} &\gamma = \rho^*/\rho, \quad \alpha_1^* = \left(1 - \frac{c^2}{\bar{c}_L^{*2}}\right)^{1/2}, \quad \alpha_2^* = \left(1 - \frac{c^2}{\bar{c}_T^{*2}}\right)^{1/2}, \\ &\bar{c}_L^{*2} = (2\bar{\mu}^* + \bar{\lambda}^*)/\rho^*, \quad \bar{c}_T^{*2} = \bar{\mu}^*/\rho^*, \\ &K = 2(\bar{c}_T^2 - \gamma\bar{c}_T^{*2}). \end{aligned} \quad (3.2a)$$

An inspection of eqn (3.2) shows that the form of this equation is identical with the form of its local classical counterpart ([14], p. 112; [15], p. 540). A fundamental difference between the two equations, however, is that the eqn (3.2) predicts a dispersion of Stoneley waves, while its local alternative does not. It seems that the following observations with regard to the eqn (3.2) may be of interest.

(1) The nonlocal character of eqn (3.2) lies in the presence of the nonlocal velocities, such as $\bar{c}_L$, $\bar{c}_T$, $\bar{c}_L^*$ and $\bar{c}_T^*$, which, on account of the relations (2.7), represent functions of the wave vector k.

(2) For long waves, that is, for $k \to 0$, eqn (3.2) transforms into the classical frequency equation derived by Stoneley[16] for the general case of the compressible media.

(3) Since the relations between the local and the nonlocal velocities are of the form

$$\bar{c}_L = \epsilon^{1/2} c_L, \tag{3.3}$$

say, where c_L is the classical velocity of the longitudinal waves, solution of the nonlocal equation (3.2) may be found directly from the corresponding solution of its local counterpart by means of the formula

$$c_{\text{nonlocal}} = \epsilon^{1/2} c_{\text{local}}, \tag{3.4}$$

without explicitly solving this equation.

(4) If the mass density ρ^*, say, is set equal to zero, eqn (3.2) transforms into the equation for the nonlocal Rayleigh surface waves derived directly in [11]. We note that the expression $[1 + \bar{\mu}(\xi)/\mu]$ in eqn (3.14) in [11] corresponds to $\epsilon(k)$ in eqn (2.7) of the present text.

(5) If one sets $\rho = \rho^*$, $\bar{\mu} = \bar{\mu}^*$, and $\bar{\lambda} = \bar{\lambda}^*$, one arrives at the case of the plane waves propagating in an infinite nonlocal medium with the velocities $c = \bar{c}_L$ and $c = \bar{c}_T$, correspondingly [compare eqns (2.13)].

(6) Since the value of $\epsilon^{1/2}$ in the Brillouin zone varies from 1 to 0.637, the velocity of very short Stoneley waves predicted by the nonlocal theory is by 36.3% less than the velocity of very long Stoneley waves found in the conventional theory:

$$c_{\text{nonlocal}}^{\text{Stoneley}} = 0.637 c_{\text{class}}^{\text{Stoneley}}. \tag{3.5}$$

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